On critical p-Laplacian systems *

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Abstract

We consider the critical p-Laplacian system

$$\begin{cases} -\Delta_p u - \frac{\lambda a}{p}|u|^{a-2}u|v|^b = \mu_1|u|^{p^*-2}u + \frac{\alpha\gamma}{p^*}|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega, \\ -\Delta_p v - \frac{\lambda b}{p}|u|^{a}|v|^{b-2}v = \mu_2|v|^{p^*-2}v + \frac{\beta\gamma}{p^*}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u, v \text{ in } D_0^{1,p}(\Omega), \end{cases} \tag{0.1}$$

where $\Delta_p := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator defined on $D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}$, endowed with norm $\|u\|_{D^{1,p}} := \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}$, $N \geq 3$, $1 < p < N$, $\lambda, \mu_1, \mu_2 \geq 0$, $\gamma \neq 0$, $a, b, \alpha, \beta > 1$ satisfy $a + b = p, \alpha + \beta = p^* := \frac{Np}{N-p}$, the critical Sobolev exponent, Ω is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$, $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\mathbb{R}^N)$. Under suitable assumptions, we establish the existence and nonexistence of a positive least energy solution of (0.1). We also consider the existence and multiplicity of nontrivial nonnegative solutions.

Key words: Nehari manifold, p-Laplacian systems, least energy solutions, critical exponent.

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1 Introduction

Equations and systems involving the p-Laplacian operator have been extensively studied in the recent years (see, e.g., [2, 3, 5, 7, 8, 9, 10, 13, 16, 17, 19, 20, 22, 23, 24] and their references). In the present paper, we study the critical p-Laplacian system

$$\begin{cases} -\Delta_p u - \frac{\lambda a}{p}|u|^{a-2}u|v|^b = \mu_1|u|^{p^*-2}u + \frac{\alpha\gamma}{p^*}|u|^{\alpha-2}u|v|^\beta, & x \in \Omega, \\ -\Delta_p v - \frac{\lambda b}{p}|u|^a|v|^{b-2}v = \mu_2|v|^{p^*-2}v + \frac{\beta\gamma}{p^*}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega, \\ u, v \text{ in } D_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator defined on $D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}$, endowed with norm $\|u\|_{D^{1,p}} := \big(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\big)^{\frac{1}{p}}$, $N \geq 3$, $1 < p < N$, $\lambda, \mu_1, \mu_2 \geq 0$, $\gamma \neq 0$, $a, b, \alpha, \beta > 1$ satisfy $a + b = p, \alpha + \beta = p^* := \frac{Np}{N-p}$, the critical Sobolev exponent, Ω is $\mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$, and $D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{1,p}(\mathbb{R}^N)$. Note that we allow the powers in the coupling terms to be unequal. We consider the two cases

$(\mathbf{H_1})$ $\Omega = \mathbb{R}^N, \lambda = 0, \mu_1, \mu_2 > 0$;

$(\mathbf{H_2})$ Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0, \mu_1, \mu_2 = 0, \gamma = 1$.

Let

$$S := \inf_{u \in D_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \mathrm{d}x}{\Big(\int_\Omega |u|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}} \tag{1.2}$$

be the sharp constant of imbedding for $D_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ (see, e.g., [1]). Then S is independent of Ω and is attained only when $\Omega = \mathbb{R}^N$. In this case a minimizer $u \in D^{1,p}(\mathbb{R}^N)$ satisfies the critical p-Laplacian equation

$$-\Delta_p u = |u|^{p^*-2}u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

Damascelli et al. [14] recently showed that all solutions of (1.3) are radial and radially decreasing about some point in $\mathbb{R}^N$ when $1 < p < 2 \leq p^*$. Sciunzi [21] extended this result to the case $2 < p < N$. By exploiting the classification results in [4, 18], we see that, for $1 < p < N$, all positive solutions of (1.3) are of the form

$$U_{\varepsilon,y}(x) := \Big[N\Big(\frac{N-p}{p-1}\Big)^{p-1}\Big]^{\frac{N-p}{p^2}} \Big(\frac{\varepsilon^{\frac{1}{p-1}}}{\varepsilon^{\frac{p}{p-1}} + |x-y|^{\frac{p}{p-1}}}\Big)^{\frac{N-p}{p}}, \quad \varepsilon > 0, \, y \in \mathbb{R}^N, \tag{1.4}$$

and

$$\int_{\mathbb{R}^N} |\nabla U_{\varepsilon,y}|^p \mathrm{d}x = \int_{\mathbb{R}^N} |U_{\varepsilon,y}|^{p^*} \mathrm{d}x = S^{\frac{N}{p}}. \tag{1.5}$$

In the case (H_1), the energy functional associated with the system (1.1) is given by

$$I(u,v)=\frac{1}{p}\int_{\mathbb{R}^N}\big(|\nabla u|^p+|\nabla v|^p\big)-\frac{1}{p^*}\int_{\mathbb{R}^N}\big(\mu_1|u|^{p^*}+\mu_2|v|^{p^*}+\gamma|u|^\alpha|v|^\beta\big),$$
$$(u,v)\in D, \quad (1.6)$$

where $D:=D^{1,p}(\mathbb{R}^N)\times D^{1,p}(\mathbb{R}^N)$, endowed with norm $\|(u,v)\|_D^p=\|u\|_{D^{1,p}}^p+\|v\|_{D^{1,p}}^p$. In this case, (1.1) with $\alpha=\beta$ and $p=2$ has been studied by Chen and Zou [11, 12]. Define

$$\mathcal{N}=\Big\{(u,v)\in D: u\neq 0, v\neq 0, \int_{\mathbb{R}^N}|\nabla u|^p=\int_{\mathbb{R}^N}\big(\mu_1|u|^{p^*}+\frac{\alpha\gamma}{p^*}|u|^\alpha|v|^\beta\big),$$
$$\int_{\mathbb{R}^N}|\nabla v|^p=\int_{\mathbb{R}^N}\big(\mu_2|v|^{p^*}+\frac{\beta\gamma}{p^*}|u|^\alpha|v|^\beta\big)\Big\}.$$

It is easy to see that $\mathcal{N}\neq\emptyset$ and that any nontrivial solution of (1.1) is in $\mathcal{N}$. By a nontrivial solution we mean a solution (u,v) such that $u\neq 0$ and $v\neq 0$. A solution is called a least energy solution if its energy is minimal among energies of all nontrivial solutions. A solution (u,v) is positive if $u>0$ and $v>0$, and semitrivial if it is of the form $(u,0)$ with $u\neq 0$, or $(0,v)$ with $v\neq 0$. Set $A:=\inf_{(u,v)\in\mathcal{N}} I(u,v)$, and note that

$$A=\inf_{(u,v)\in\mathcal{N}}\frac{1}{N}\int_{\mathbb{R}^N}\big(|\nabla u|^p+|\nabla v|^p\big)$$
$$=\inf_{(u,v)\in\mathcal{N}}\frac{1}{N}\int_{\mathbb{R}^N}\big(\mu_1|u|^{p^*}+\mu_2|v|^{p^*}+\gamma|u|^\alpha|v|^\beta\big).$$

Consider the nonlinear system of equations

$$\begin{cases}\mu_1 k^{\frac{p^*-p}{p}}+\frac{\alpha\gamma}{p^*}k^{\frac{\alpha-p}{p}}l^{\frac{\beta}{p}}=1,\\ \mu_2 l^{\frac{p^*-p}{p}}+\frac{\beta\gamma}{p^*}k^{\frac{\alpha}{p}}l^{\frac{\beta-p}{p}}=1,\\ k>0,\ l>0.\end{cases} \quad (1.7)$$

Our main results in this case are the following.

Theorem 1.1. *If (H_1) holds and $\gamma<0$, then $A=\frac{1}{N}\big(\mu_1^{-\frac{N-p}{p}}+\mu_2^{-\frac{N-p}{p}}\big)S^{\frac{N}{p}}$ and A is not attained.*

Theorem 1.2. *If (H_1) and*

(C_1) $\frac{N}{2}<p<N, \alpha,\beta>p$, *and*

$$0<\gamma\leq\frac{3p^2}{(3-p)^2}\min\Big\{\frac{\mu_1}{\alpha}\Big(\frac{\alpha-p}{\beta-p}\Big)^{\frac{\beta-p}{p}},\ \frac{\mu_2}{\beta}\Big(\frac{\beta-p}{\alpha-p}\Big)^{\frac{\alpha-p}{p}}\Big\} \quad (1.8)$$

or

(C_2) $\frac{2N}{N+2} < p < \frac{N}{2}, \alpha, \beta < p$, *and*

$$\gamma \geq \frac{Np^2}{(N-p)^2} \max\Big\{ \frac{\mu_1}{\alpha}\Big(\frac{p-\beta}{p-\alpha}\Big)^{\frac{p-\beta}{p}},\ \frac{\mu_2}{\beta}\Big(\frac{p-\alpha}{p-\beta}\Big)^{\frac{p-\alpha}{p}} \Big\} \tag{1.9}$$

hold, then $A = \frac{1}{N}(k_0+l_0)S^{\frac{N}{p}}$ *and* A *is attained by* $(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y})$, *where* (k_0, l_0) *satisfies (1.7) and*

$$k_0 = \min\{k : (k,l) \text{ satisfies (1.7)}\}. \tag{1.10}$$

Theorem 1.3. *Assume that* $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$, *and* (H_1) *holds. If* $\gamma > 0$, *then* A *is attained by some* (U, V), *where* U *and* V *are positive, radially symmetric, and decreasing.*

Theorem 1.4. *(Multiplicity) Assume that* $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$, *and* (H_1) *holds. There exists*

$$\gamma_1 \in \Big(0, \frac{Np^2}{(N-p)^2} \max\Big\{ \frac{\mu_1}{\alpha}\Big(\frac{2-\beta}{2-\alpha}\Big)^{\frac{2-\beta}{2}},\ \frac{\mu_2}{\beta}\Big(\frac{2-\alpha}{2-\beta}\Big)^{\frac{2-\alpha}{2}} \Big\}\Big]$$

such that for any $\gamma \in (0, \gamma_1)$, *there exists a solution* $\big(k(\gamma), l(\gamma)\big)$ *of (1.7) satisfying*

$$I\big(\sqrt[p]{k(\gamma)}U_{\varepsilon,y}, \sqrt[p]{l(\gamma)}U_{\varepsilon,y}\big) > A$$

and $\big(\sqrt[p]{k(\gamma)}U_{\varepsilon,y}, \sqrt[p]{l(\gamma)}U_{\varepsilon,y}\big)$ *is a (second) positive solution of (1.1).*

For the case (H_2), we have the following theorem.

Theorem 1.5. *If* (H_2) *holds,* $p \leq \sqrt{N}$, *and*

$$0 < \lambda < \frac{p}{(a^a\, b^b)^{\frac{1}{p}}}\, \lambda_1(\Omega),$$

where $\lambda_1(\Omega) > 0$ *is the first Dirichlet eigenvalue of* $-\Delta_p$ *in* Ω, *then the system (1.1) has a nontrivial nonnegative solution.*

2 Proof of Theorem 1.1

Lemma 2.1. *Assume that* (H_1) *holds and* $-\infty < \gamma < 0$. *If* A *is attained by a couple* $(u, v) \in \mathcal{N}$, *then* (u, v) *is a critical point of* I, *i.e.,* (u, v) *is a solution of (1.1).*

Proof. Define

$$\begin{aligned}
\mathcal{N}_1 := \Big\{&(u,v) \in D :\ u \not\equiv 0,\ v \not\equiv 0,\\
&G_1(u,v) := \int_{\mathbb{R}^N} |\nabla u|^p - \int_{\mathbb{R}^N} \big(\mu_1|u|^{p^*} + \frac{\alpha\gamma}{p^*}|u|^\alpha|v|^\beta\big) = 0\Big\},\\
\mathcal{N}_2 := \Big\{&(u,v) \in D :\ u \not\equiv 0,\ v \not\equiv 0,\\
&G_2(u,v) := \int_{\mathbb{R}^N} |\nabla v|^p - \int_{\mathbb{R}^N} \big(\mu_2|v|^{p^*} + \frac{\beta\gamma}{p^*}|u|^\alpha|v|^\beta\big) = 0\Big\}.
\end{aligned}$$

Obviously, $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$. Suppose that $(u, v) \in \mathcal{N}$ is a minimizer for I restricted to $\mathcal{N}$. It follows from the standard minimization theory that there exist two Lagrange multipliers $L_1, L_2 \in \mathbb{R}$ such that

$$I'(u,v) + L_1 G_1'(u,v) + L_2 G_2'(u,v) = 0.$$

Noticing that

$$\begin{aligned}
I'(u,v)(u,0) &= G_1(u,v) = 0,\\
I'(u,v)(0,v) &= G_2(u,v) = 0,\\
G_1'(u,v)(u,0) &= -(p^*-p)\int_{\mathbb{R}^N}\mu_1|u|^{p^*} + (p-\alpha)\int_{\mathbb{R}^N}\frac{\alpha\gamma}{p^*}|u|^\alpha|v|^\beta,\\
G_1'(u,v)(0,v) &= -\beta\int_{\mathbb{R}^N}\frac{\alpha\gamma}{p^*}|u|^\alpha|v|^\beta > 0,\\
G_2'(u,v)(u,0) &= -\alpha\int_{\mathbb{R}^N}\frac{\beta\gamma}{p^*}|u|^\alpha|v|^\beta > 0,\\
G_2'(u,v)(0,v) &= -(p^*-p)\int_{\mathbb{R}^N}\mu_2|v|^{p^*} + (p-\beta)\int_{\mathbb{R}^N}\frac{\beta\gamma}{p^*}|u|^\alpha|v|^\beta,
\end{aligned}$$

we get that

$$\begin{cases} G_1'(u,v)(u,0)L_1 + G_2'(u,v)(u,0)L_2 = 0,\\ G_1'(u,v)(0,v)L_1 + G_2'(u,v)(0,v)L_2 = 0,\end{cases}$$

and

$$\begin{aligned}
G_1'(u,v)(u,0) + G_1'(u,v)(0,v) &= -(p^*-p)\int_{\mathbb{R}^N}|\nabla u|^p \le 0,\\
G_2'(u,v)(u,0) + G_2'(u,v)(0,v) &= -(p^*-p)\int_{\mathbb{R}^N}|\nabla v|^p \le 0.
\end{aligned}$$

We claim that $\int_{\mathbb{R}^N}|\nabla u|^p > 0$. Indeed, if $\int_{\mathbb{R}^N}|\nabla u|^p = 0$, then by (1.2), we have $\int_{\mathbb{R}^N}|u|^{p^*} \le S^{-\frac{p^*}{p}}\left(\int_{\mathbb{R}^N}|\nabla u|^p\right)^{\frac{p^*}{p}} = 0$. Thus, a desired contradiction comes out, $u \equiv 0$ almost everywhere in $\mathbb{R}^N$. Similarly, $\int_{\mathbb{R}^N}|\nabla v|^p > 0$. Hence,

$$\begin{aligned}
\left|G_1'(u,v)(u,0)\right| &= -G_1'(u,v)(u,0) > G_1'(u,v)(0,v),\\
\left|G_2'(u,v)(0,v)\right| &= -G_2'(u,v)(0,v) > G_2'(u,v)(u,0).
\end{aligned}$$

Define the matrix

$$M := \begin{pmatrix} G_1'(u,v)(u,0) & G_2'(u,v)(u,0)\\ G_1'(u,v)(0,v) & G_2'(u,v)(0,v)\end{pmatrix},$$

then,

$$\begin{aligned}
\det(M) =& \left|G_1'(u,v)(u,0)\right| \cdot \left|G_2'(u,v)(0,v)\right|\\
&- G_1'(u,v)(0,v)\cdot G_2'(u,v)(u,0) > 0,
\end{aligned}$$

which means that $L_1 = L_2 = 0$. This completes the proof. □

Proof of Theorem 1.1. It is standard to see that $A > 0$. By (1.4), we know that $\omega_{\mu_i} := \mu_i^{\frac{p-N}{p^2}} U_{1,0}$ satisfies $-\Delta_p u = \mu_i |u|^{p^*-2}u$ in $\mathbb{R}^N$, where $i = 1, 2$. Set $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and

$$\big(u_R(x), v_R(x)\big) = \big(\omega_{\mu_1}(x), \omega_{\mu_2}(x + Re_1)\big),$$

where R is a positive number. Then, $v_R \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $v_R \rightharpoonup 0$ weakly in $L^{p^*}(\mathbb{R}^N)$ as $R \to +\infty$. Hence,

$$\begin{aligned}
\lim_{R\to+\infty} \int_{\mathbb{R}^N} u_R^\alpha v_R^\beta \mathrm{d}x &= \lim_{R\to+\infty} \int_{\mathbb{R}^N} u_R^\alpha v_R^{\frac{\alpha}{p^*-1}} v_R^{\frac{p^*(\beta-1)}{p^*-1}} \mathrm{d}x \\
&\le \lim_{R\to+\infty} \Big(\int_{\mathbb{R}^N} u_R^{p^*-1} v_R \mathrm{d}x \Big)^{\frac{\alpha}{p^*-1}} \Big(\int_{\mathbb{R}^N} v_R^{p^*} \mathrm{d}x \Big)^{\frac{\beta-1}{p^*-1}} \\
&= 0.
\end{aligned}$$

Therefore, for $R > 0$ sufficiently large, the system

$$\begin{cases}
\int_{\mathbb{R}^N} |\nabla u_R|^p \mathrm{d}x = \int_{\mathbb{R}^N} \mu_1 u_R^{p^*} \mathrm{d}x \\
\qquad = t_R^{\frac{p^*-p}{p}} \int_{\mathbb{R}^N} \mu_1 u_R^{p^*} \mathrm{d}x + t_R^{\frac{\alpha-p}{p}} s_R^{\frac{\beta}{p}} \int_{\mathbb{R}^N} \frac{\alpha\gamma}{p^*} u_R^\alpha v_R^\beta \mathrm{d}x, \\
\int_{\mathbb{R}^N} |\nabla v_R|^p \mathrm{d}x = \int_{\mathbb{R}^N} \mu_2 v_R^{p^*} \mathrm{d}x \\
\qquad = s_R^{\frac{p^*-p}{p}} \int_{\mathbb{R}^N} \mu_2 v_R^{p^*} \mathrm{d}x + t_R^{\frac{\alpha}{p}} s_R^{\frac{\beta-p}{p}} \int_{\mathbb{R}^N} \frac{\beta\gamma}{p^*} u_R^\alpha v_R^\beta \mathrm{d}x
\end{cases}$$

has a solution (t_R, s_R) with

$$\lim_{R\to+\infty} \big(|t_R - 1| + |s_R - 1|\big) = 0.$$

Furthermore, $(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R) \in \mathcal{N}$. Then, by (1.5), we obtain that

$$\begin{aligned}
A &= \inf_{(u,v)\in\mathcal{N}} I(u, v) \le I(\sqrt[p]{t_R} u_R, \sqrt[p]{s_R} v_R) \\
&= \frac{1}{N}\Big(t_R \int_{\mathbb{R}^N} |\nabla u_R|^p \mathrm{d}x + s_R \int_{\mathbb{R}^N} |\nabla v_R|^p \mathrm{d}x \Big) \\
&= \frac{1}{N}\big(t_R \mu_1^{-\frac{N-p}{p}} + s_R \mu_2^{-\frac{N-p}{p}} \big) S^{\frac{N}{p}},
\end{aligned}$$

which implies that $A \le \frac{1}{N}\big(\mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}}\big) S^{\frac{N}{p}}$.

For any $(u, v) \in \mathcal{N}$,

$$\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \le \mu_1 \int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x \le \mu_1 S^{-\frac{p^*}{p}} \Big(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \Big)^{\frac{p^*}{p}}.$$

Therefore, $\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \ge \mu_1^{-\frac{N-p}{p}} S^{\frac{N}{p}}$. Similarly, $\int_{\mathbb{R}^N} |\nabla v|^p \mathrm{d}x \ge \mu_2^{-\frac{N-p}{p}} S^{\frac{N}{p}}$. Then, $A \ge \frac{1}{N}\big(\mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}}\big) S^{\frac{N}{p}}$. Hence,

$$A = \frac{1}{N}\big(\mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}}\big) S^{\frac{N}{p}}. \tag{2.1}$$

Suppose by contradiction that A is attained by some $(u,v)\in\mathcal{N}$. Then $(|u|,|v|)\in\mathcal{N}$ and $I(|u|,|v|)=A$. By Lemma 2.1, we see that $(|u|,|v|)$ is a nontrivial solution of (1.1). By strong maximum principle, we may assume that $u>0, v>0$, and so $\int_{\mathbb{R}^N} u^\alpha v^\beta \mathrm{d}x>0$. Then,

$$\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x<\mu_1\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x\le\mu_1 S^{-\frac{p^*}{p}}\Big(\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x\Big)^{\frac{p^*}{p}},$$

which yields that $\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x>\mu_1^{-\frac{N-p}{p}}S^{\frac{N}{p}}$. Similarly, $\int_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x>\mu_2^{-\frac{N-p}{p}}S^{\frac{N}{p}}$. Therefore,

$$A=I(u,v)=\frac{1}{N}\int_{\mathbb{R}^N}\big(|\nabla u|^p+|\nabla v|^p\big)\mathrm{d}x>\frac{1}{N}\big(\mu_1^{-\frac{N-p}{p}}+\mu_2^{-\frac{N-p}{p}}\big)S^{\frac{N}{p}},$$

which contradicts to (2.1). This completes the proof. □

3 Proof of Theorem 1.2

Proposition 3.1. *Assume that $c,d\in\mathbb{R}$ satisfy*

$$\begin{cases}\mu_1 c^{\frac{p^*-p}{p}}+\frac{\alpha\gamma}{p^*}c^{\frac{\alpha-p}{p}}d^{\frac{\beta}{p}}\ge 1,\\ \mu_2 d^{\frac{p^*-p}{p}}+\frac{\beta\gamma}{p^*}c^{\frac{\alpha}{p}}d^{\frac{\beta-p}{p}}\ge 1,\\ c>0,\ d>0.\end{cases}\tag{3.1}$$

If $\frac{N}{2}<p<N, \alpha,\beta>p$ and (1.8) holds, then $c+d\ge k+l$, where $k,l\in\mathbb{R}$ satisfy (1.7).

Proof. Let $y=c+d, x=\frac{c}{d}, y_0=k+l$, and $x_0=\frac{k}{l}$. By (3.1) and (1.7), we have that

$$y^{\frac{p^*-p}{p}}\ge\frac{(x+1)^{\frac{p^*-p}{p}}}{\mu_1 x^{\frac{p^*-p}{p}}+\frac{\alpha\gamma}{p^*}x^{\frac{\alpha-p}{p}}}:=f_1(x),\qquad y_0^{\frac{p^*-p}{p}}=f_1(x_0),$$

$$y^{\frac{p^*-p}{p}}\ge\frac{(x+1)^{\frac{p^*-p}{p}}}{\mu_2+\frac{\beta\gamma}{p^*}x^{\frac{\alpha}{p}}}:=f_2(x),\qquad y_0^{\frac{p^*-p}{p}}=f_2(x_0).$$

Thus,

$$f_1'(x)=\frac{\alpha\gamma(x+1)^{\frac{p^*-2p}{p}}x^{\frac{\alpha-2p}{p}}}{pp^*(\mu_1 x^{\frac{p^*-p}{p}}+\frac{\alpha\gamma}{p^*}x^{\frac{\alpha-p}{p}})^2}\Big[-\frac{p^*(p^*-p)\mu_1}{\alpha\gamma}x^{\frac{\beta}{p}}+\beta x-(\alpha-p)\Big],$$

$$f_2'(x)=\frac{\beta\gamma(x+1)^{\frac{p^*-2p}{p}}}{pp^*(\mu_2+\frac{\beta\gamma}{p^*}x^{\frac{\alpha}{p}})^2}\Big[(\beta-p)x^{\frac{\alpha}{p}}-\alpha x^{\frac{\alpha-p}{p}}+\frac{p^*(p^*-p)\mu_2}{\beta\gamma}\Big].$$

Let $x_1 = \left(\frac{p\alpha\gamma}{p^*(p^*-p)\mu_1}\right)^{\frac{p}{\beta-p}}$, $x_2 = \frac{\alpha-p}{\beta-p}$ and

$$
\begin{aligned}
g_1(x) &= -\frac{p^*(p^*-p)\mu_1}{\alpha\gamma} x^{\frac{\beta}{p}} + \beta x - (\alpha - p), \\
g_2(x) &= (\beta - p) x^{\frac{\alpha}{p}} - \alpha x^{\frac{\alpha-p}{p}} + \frac{p^*(p^*-p)\mu_2}{\beta\gamma}.
\end{aligned}
$$

It follows from (1.8) that

$$
\begin{aligned}
\max_{x\in(0,+\infty)} g_1(x) = g_1(x_1) =& (\beta - p)\Big(\frac{p\alpha\gamma}{p^*(p^*-p)\mu_1}\Big)^{\frac{p}{\beta-p}} - (\alpha - p) \le 0, \\
\min_{x\in(0,+\infty)} g_2(x) = g_2(x_2) =& -p\Big(\frac{\alpha-p}{\beta-p}\Big)^{\frac{\alpha-p}{p}} + \frac{p^*(p^*-p)\mu_2}{\beta\gamma} \ge 0.
\end{aligned}
$$

That is, $f_1(x)$ is strictly decreasing in $(0, +\infty)$ and $f_2(x)$ is strictly increasing in $(0, +\infty)$. Hence,

$$
\begin{aligned}
y^{\frac{p^*-p}{p}} \ge \max\{f_1(x), f_2(x)\} \ge& \min_{x\in(0,+\infty)} \Big(\max\{f_1(x), f_2(x)\}\Big) \\
&= \min_{\{f_1=f_2\}} \Big(\max\{f_1(x), f_2(x)\}\Big) = y_0^{\frac{p^*-p}{p}},
\end{aligned}
$$

where $\{f_1 = f_2\} := \{x \in (0, +\infty) : f_1(x) = f_2(x)\}$. This completes the proof. □

Remark 3.1. *From the proof of Proposition 3.1, it is easy to see that the system (1.7), under the assumption of Proposition 3.1, has only one real solution* $(k, l) = (k_0, l_0)$*, where* (k_0, l_0) *is defined as in (1.10).*

Define functions:

$$
\begin{aligned}
F_1(k,l) &:= \mu_1 k^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} k^{\frac{\alpha-p}{p}} l^{\frac{\beta}{p}} - 1, \;\; k > 0, l \ge 0; \\
F_2(k,l) &:= \mu_2 l^{\frac{p^*-p}{p}} + \frac{\beta\gamma}{p^*} k^{\frac{\alpha}{p}} l^{\frac{\beta-p}{p}} - 1, \;\; k \ge 0, l > 0; \\
l(k) &:= \Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{p}{\beta}} k^{\frac{p-\alpha}{\beta}} \big(1 - \mu_1 k^{\frac{p^*-p}{p}}\big)^{\frac{p}{\beta}}, \;\; 0 < k \le \mu_1^{-\frac{p}{p^*-p}}; \\
k(l) &:= \Big(\frac{p^*}{\beta\gamma}\Big)^{\frac{p}{\alpha}} l^{\frac{p-\beta}{\alpha}} \big(1 - \mu_2 l^{\frac{p^*-p}{p}}\big)^{\frac{p}{\alpha}}, \;\; 0 < l \le \mu_2^{-\frac{p}{p^*-p}}.
\end{aligned}
\tag{3.2}
$$

Then, $F_1\big(k, l(k)\big) \equiv 0$ and $F_2\big(k(l), l\big) \equiv 0$.

Lemma 3.1. *Assume that* $\frac{2N}{N+2} < p < \frac{N}{2}, \alpha, \beta < p, \gamma > 0$*. Then*

$$
F_1(k,l) = 0, \;\; F_2(k,l) = 0, \;\; k, l > 0 \tag{3.3}
$$

has a solution (k_0, l_0) *such that*

$$
F_2(k, l(k)) < 0, \;\; \forall k \in (0, k_0), \tag{3.4}
$$

that is, (k_0, l_0) *satisfies (1.10). Similarly, (3.3) has a solution* (k_1, l_1) *such that*

$$F_1(k(l), l) < 0, \ \ \forall l \in (0, l_1), \tag{3.5}$$

that is,

$$(k_1, l_1) \textit{ satisfies (1.7) and } l_1 = \min\{l : (k, l) \textit{ is a solution of (1.7)}\}. \tag{3.6}$$

Proof. We only prove the existence of (k_0, l_0). It follows from $F_1(k, l) = 0, \ k, l > 0$ that

$$l = l(k), \ \ \forall k \in (0, \mu_1^{-\frac{p}{p^*-p}}).$$

Substituting this into $F_2(k, l) = 0$, we have

$$\begin{aligned} &\mu_2\Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{\alpha}{\beta}}\Big(1-\mu_1 k^{\frac{p^*-p}{p}}\Big)^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{p^*}k^{\frac{(p^*-p)\alpha}{p\beta}} \\ &- \Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{p-\beta}{\beta}} k^{-\frac{(p^*-p)(p-\alpha)}{p\beta}}\Big(1-\mu_1 k^{\frac{p^*-p}{p}}\Big)^{\frac{p-\beta}{\beta}} = 0. \end{aligned} \tag{3.7}$$

Setting

$$\begin{aligned} f(k) :=&\mu_2\Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{\alpha}{\beta}}\Big(1-\mu_1 k^{\frac{p^*-p}{p}}\Big)^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{p^*}k^{\frac{(p^*-p)\alpha}{p\beta}} \\ &- \Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{p-\beta}{\beta}} k^{-\frac{(p^*-p)(p-\alpha)}{p\beta}}\Big(1-\mu_1 k^{\frac{p^*-p}{p}}\Big)^{\frac{p-\beta}{\beta}}, \end{aligned} \tag{3.8}$$

then the existence of a solution of (3.7) in $(0, \mu_1^{-\frac{p}{p^*-p}})$ is equivalent to $f(k) = 0$ possessing a solution in $(0, \mu_1^{-\frac{p}{p^*-p}})$. Since $\alpha, \beta < p$, we get that

$$\lim_{k\to 0^+} f(k) = -\infty, \qquad f\big(\mu_1^{-\frac{p}{p^*-p}}\big) = \frac{\beta\gamma}{p^*}\mu_1^{-\frac{\alpha}{\beta}} > 0,$$

which implies that there exists $k_0 \in \big(0, \mu_1^{-\frac{p}{p^*-p}}\big)$ such that $f(k_0) = 0$ and $f(k) < 0$ for $k \in (0, k_0)$. Let $l_0 = l(k_0)$. Then (k_0, l_0) is a solution of (3.3) and (3.4) holds. ☐

Remark 3.2. *From* $\frac{2N}{N+2} < p < \frac{N}{2}$ *and* $\alpha, \beta < p$*, we get that* $2 < p^* < 2p$*. It can be seen from* $\frac{N}{2} < p < N$ *and* $\alpha, \beta > p$ *that* $2 < 2p < p^*$*.*

Lemma 3.2. *Assume that* $\frac{2N}{N+2} < p < \frac{N}{2}, \alpha, \beta < p$*, and (1.9) holds. Let* (k_0, l_0) *be the same as in Lemma 3.1. Then,*

$$(k_0 + l_0)^{\frac{p^*-p}{p}} \max\{\mu_1, \mu_2\} < 1 \tag{3.9}$$

and

$$F_2\big(k, l(k)\big) < 0, \ \ \forall k \in (0, k_0); \ \ F_1\big(k(k), l\big) < 0, \ \ \forall l \in (0, l_0). \tag{3.10}$$

Proof. Recalling (3.2), we obtain that

$$
\begin{aligned}
l'(k) =&\Big(\frac{p^*}{\alpha\gamma}\Big)^{\frac{p}{\beta}}\frac{p}{\beta}\Big(k^{\frac{p-\alpha}{p}}-\mu_1 k^{\frac{\beta}{p}}\Big)^{\frac{p-\beta}{\beta}}\Big(\frac{p-\alpha}{p}k^{-\frac{\alpha}{p}}-\frac{\mu_1\beta}{p}k^{\frac{\beta-p}{p}}\Big)\\
=&\Big(\frac{p^*\mu_1}{\alpha\gamma}\Big)^{\frac{p}{\beta}}k^{\frac{p-p^*}{\beta}}\Big(\mu_1^{-1}-k^{\frac{p^*-p}{p}}\Big)^{\frac{p-\beta}{\beta}}\Big(\frac{p-\alpha}{\mu_1\beta}-k^{\frac{p^*-p}{p}}\Big),
\end{aligned}
$$

$l'\big((\frac{p-\alpha}{\mu_1\beta})^{\frac{p}{p^*-p}}\big)=l'\big(\mu_1^{-\frac{p}{p^*-p}}\big)=0$, $l'(k)>0$ for $k\in\big(0,(\frac{p-\alpha}{\mu_1\beta})^{\frac{p}{p^*-p}}\big)$, and $l'(k)<0$ for $k\in\big((\frac{p-\alpha}{\mu_1\beta})^{\frac{p}{p^*-p}},\mu_1^{-\frac{p}{p^*-p}}\big)$. From

$$
\begin{aligned}
l''(\bar k) =&\frac{p-\beta}{\beta}\Big(\frac{p^*\mu_1}{\alpha\gamma}\Big)^{\frac{p}{\beta}}\bar k^{\frac{p-2\beta-\alpha}{\beta}}\Big(\mu_1^{-1}-\bar k^{\frac{p^*-p}{p}}\Big)^{\frac{p-2\beta}{\beta}}\\
&\cdot\Big[\Big(\frac{p-\alpha}{\mu_1\beta}-\bar k^{\frac{p^*-p}{p}}\Big)^2-\Big(\mu_1^{-1}-\bar k^{\frac{p^*-p}{p}}\Big)\Big(\frac{\alpha(p-\alpha)}{\mu_1\beta(p-\beta)}-\bar k^{\frac{p^*-p}{p}}\Big)\Big]=0
\end{aligned}
$$

and $\bar k\in\big((\frac{p-\alpha}{\mu_1\beta})^{\frac{p}{p^*-p}},\mu_1^{-\frac{p}{p^*-p}}\big)$, we have $\bar k=\big(\frac{p(p-\alpha)}{(2p-p^*)\mu_1\beta}\big)^{\frac{p}{p^*-p}}$. Then, by (1.9), we get that

$$
\begin{aligned}
\min_{k\in\big(0,\mu_1^{-\frac{p}{p^*-p}}\big]} l'(k)=&\min_{k\in\big((\frac{p-\alpha}{\mu_1\beta})^{\frac{p}{p^*-p}},\mu_1^{-\frac{p}{p^*-p}}\big]} l'(k)=l'(\bar k)\\
=&-\Big(\frac{p^*(p^*-p)\mu_1}{p\alpha\gamma}\Big)^{\frac{p}{\beta}}\Big(\frac{p-\beta}{p-\alpha}\Big)^{\frac{p-\beta}{\beta}}\\
\geq&-1.
\end{aligned}
$$

Therefore, $l'(k)>-1$ for $k\in\big(0,\mu_1^{-\frac{p}{p^*-p}}\big]$ with $k\neq\big(\frac{p(p-\alpha)}{(2p-p^*)\mu_1\beta}\big)^{\frac{p}{p^*-p}}$, which implies that $l(k)+k$ is strictly increasing on $\big[0,\mu_1^{-\frac{p}{p^*-p}}\big]$. Noticing that $k_0<\mu_1^{-\frac{p}{p^*-p}}$, we have

$$
\mu_1^{-\frac{p}{p^*-p}}=l\big(\mu_1^{-\frac{p}{p^*-p}}\big)+\mu_1^{-\frac{p}{p^*-p}}>l(k_0)+k_0=l_0+k_0,
$$

that is, $\mu_1(k_0+l_0)^{\frac{p^*-p}{p}}<1$. Similarly, $\mu_2(k_0+l_0)^{\frac{p^*-p}{p}}<1$. To prove (3.10), by Lemma 3.1, it suffices to show that $(k_0,l_0)=(k_1,l_1)$. It follows from (3.4) and (3.5) that $k_1\geq k_0$ and $l_0\geq l_1$. Suppose by contradiction that $k_1>k_0$. Then $l(k_1)+k_1>l(k_0)+k_0$. Hence, $l_1+k(l_1)=l(k_1)+k_1>l(k_0)+k_0=l_0+k(l_0)$. Following the arguments in the beginning of the proof, we have $l+k(l)$ is strictly increasing for $l\in\big[0,\mu_2^{-\frac{p}{p^*-p}}\big]$. Therefore, $l_1>l_0$, which contradicts to $l_0\geq l_1$. Then, $k_1=k_0$, and similarly, $l_0=l_1$. This completes the proof. □

Remark 3.3. *For any $\gamma>0$, the condition (1.9) always holds for the dimension N large enough.*

Proposition 3.2. *Assume that $\frac{2N}{N+2}<p<\frac{N}{2},\alpha,\beta<p$, and (1.9) holds. Then*

$$
\begin{cases}
k+l\leq k_0+l_0,\\
F_1(k,l)\geq 0,\ \ F_2(k,l)\geq 0,\\
k,l\geq 0,\ \ (k,l)\neq(0,0)
\end{cases}
\tag{3.11}
$$

has an unique solution $(k, l) = (k_0, l_0)$.

Proof. Obviously, (k_0, l_0) satisfies (3.11). Suppose that $(\tilde{k}, \tilde{l})$ is any solution of (3.11), and without loss of generality, assume that $\tilde{k} > 0$. We claim that $\tilde{l} > 0$. In fact, if $\tilde{l} = 0$, then $\tilde{k} \leq k_0 + l_0$ and $F_1(\tilde{k}, 0) = \mu_1 \tilde{k}^{\frac{p^*-p}{p}} - 1 \geq 0$. Thus,

$$1 \leq \mu_1 \tilde{k}^{\frac{p^*-p}{p}} \leq \mu_1 (k_0 + l_0)^{\frac{p^*-p}{p}},$$

a contradiction with Lemma 3.2.

Suppose by contradiction that $\tilde{k} < k_0$. It can be seen that $k(l)$ is strictly increasing on $\big(0, (\frac{p-\beta}{\mu_2 \alpha})^{\frac{p}{p^*-p}}\big]$ and strictly decreasing on $\big[(\frac{2-\beta}{\mu_2 \alpha})^{\frac{p}{p^*-p}}, \mu_2^{-\frac{p}{p^*-p}}\big]$, and $k(0) = k\big(\mu_2^{-\frac{p}{p^*-p}}\big) = 0$. Since $0 < \tilde{k} < k_0 = k(l_0)$, there exist $0 < l_1 < l_2 < \mu_2^{-\frac{p}{p^*-p}}$ such that $k(l_1) = k(l_2) = \tilde{k}$ and

$$F_2(\tilde{k}, l) < 0 \Longleftrightarrow \tilde{k} < k(l) \Longleftrightarrow l_1 < l < l_2. \tag{3.12}$$

It follows from $F_1(\tilde{k}, \tilde{l}) \geq 0$ and $F_2(\tilde{k}, \tilde{l}) \geq 0$ that $\tilde{l} \geq l(\tilde{k})$ and $\tilde{l} \leq l_1$ or $\tilde{l} \geq l_2$. By (3.10), we see $F_2\big(\tilde{k}, l(\tilde{k})\big) < 0$. By (3.12), we get that $l_1 < l(\tilde{k}) < l_2$. Therefore, $\tilde{l} \geq l_2$.

On the other hand, set $l_3 := k_0 + l_0 - \tilde{k}$. Then, $l_3 > l_0$ and moreover,

$$k(l_3) + k_0 + l_0 - \tilde{k} = k(l_3) + l_3 > k(l_0) + l_0 = k_0 + l_0,$$

that is, $k(l_3) > \tilde{k}$. By (3.12), we have $l_1 < l_3 < l_2$. Since $\tilde{k} + \tilde{l} \leq k_0 + l_0$, we obtain that $\tilde{l} \leq k_0 + l_0 - \tilde{k} = l_3 < l_2$. This contradicts to $\tilde{l} \geq l_2$, which completes the proof. $\square$

Proof of Theorem 1.2. Recalling (1.4) and (1.7), we see that $(\sqrt[p]{k_0} U_{\varepsilon,y}, \sqrt[p]{l_0} U_{\varepsilon,y}) \in \mathcal{N}$ is a nontrivial solution of (1.1), and

$$A \leq I(\sqrt[p]{k_0} U_{\varepsilon,y}, \sqrt[p]{l_0} U_{\varepsilon,y}) = \frac{1}{N}(k_0 + l_0) S^{\frac{N}{p}}. \tag{3.13}$$

Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequence for A, i.e., $I(u_n, v_n) \to A$, as $n \to \infty$. Define $c_n = (\int_{\mathbb{R}^N} |u_n|^{p^*} \mathrm{d}x)^{\frac{p}{p^*}}$ and $d_n = (\int_{\mathbb{R}^N} |v_n|^{p^*} \mathrm{d}x)^{\frac{p}{p^*}}$. Then,

$$\begin{aligned} Sc_n &\leq \int_{\mathbb{R}^N} |\nabla u_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} \big(\mu_1 |u_n|^{p^*} + \frac{\alpha\gamma}{p^*} |u_n|^{\alpha} |v_n|^{\beta}\big) \mathrm{d}x \\ &\leq \mu_1 c_n^{\frac{p^*}{p}} + \frac{\alpha\gamma}{p^*} c_n^{\frac{\alpha}{p}} d_n^{\frac{\beta}{p}}, \\ Sd_n &\leq \int_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} \big(\mu_2 |v_n|^{p^*} + \frac{\beta\gamma}{p^*} |u_n|^{\alpha} |v_n|^{\beta}\big) \mathrm{d}x \\ &\leq \mu_2 d_n^{\frac{p^*}{p}} + \frac{\beta\gamma}{p^*} c_n^{\frac{\alpha}{p}} d_n^{\frac{\beta}{p}}. \end{aligned} \tag{3.14}$$

Dividing both sides of the inequalities by Sc_n and Sd_n, respectively, and denoting $\tilde{c}_n = \frac{c_n}{S^{\frac{p}{p^*-p}}}, \tilde{d}_n = \frac{d_n}{S^{\frac{p}{p^*-p}}}$, we deduce that

$$\begin{cases} \mu_1 \tilde{c}_n^{\frac{p^*-p}{p}} + \frac{\alpha\gamma}{p^*} \tilde{c}_n^{\frac{\alpha-p}{p}} \tilde{d}_n^{\frac{\beta}{p}} \geq 1, \\ \mu_2 \tilde{d}_n^{\frac{p^*-p}{p}} + \frac{\beta\gamma}{p^*} \tilde{c}_n^{\frac{\alpha}{p}} \tilde{d}_n^{\frac{\beta-p}{p}} \geq 1, \end{cases}$$

that is, $F_1(\tilde{c}_n, \tilde{d}_n) \geq 0$ and $F_2(\tilde{c}_n, \tilde{d}_n) \geq 0$. Then, for $\frac{N}{2} < p < N$, $\alpha, \beta > p$, Proposition 3.1 and Remark 3.1 ensure that $\tilde{c}_n + \tilde{d}_n \geq k + l = k_0 + l_0$; for $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$, Proposition 3.2 guarantees that $\tilde{c}_n + \tilde{d}_n \geq k_0 + l_0$. Therefore,

$$c_n + d_n \geq (k_0 + l_0) S^{\frac{p}{p^*-p}} = (k_0 + l_0) S^{\frac{N-p}{p}}. \tag{3.15}$$

Noticing that $I(u_n, v_n) = \frac{1}{N} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + |\nabla v_n|^p\right)$, by (3.13) and (3.14), we have

$$S(c_n + d_n) \leq NI(u_n, v_n) = NA + o(1) \leq (k_0 + l_0) S^{\frac{N}{p}} + o(1).$$

Combining this with (3.15), we get that $c_n + d_n \to (k_0 + l_0) S^{\frac{N-p}{p}}$ as $n \to \infty$. Thus,

$$A = \lim_{n\to\infty} I(u_n, v_n) \geq \lim_{n\to\infty} \frac{1}{N} S(c_n + d_n) = \frac{1}{N} (k_0 + l_0) S^{\frac{N}{p}}.$$

Hence,

$$A = \frac{1}{N} (k_0 + l_0) S^{\frac{N}{p}} = I(\sqrt[p]{k_0} U_{\varepsilon,y}, \sqrt[p]{l_0} U_{\varepsilon,y}). \tag{3.16}$$

□

4 Proof of Theorems 1.3 and 1.4

For (H_1) holding and $\gamma > 0$, define

$$A' := \inf_{(u,v)\in\mathcal{N}'} I(u, v), \tag{4.1}$$

where

$$\begin{aligned} \mathcal{N}' := \Big\{ (u, v) \in D \setminus \{(0, 0)\} : & \int_{\mathbb{R}^N} \left(|\nabla u|^p + |\nabla v|^p\right) \\ & = \int_{\mathbb{R}^N} \left(\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta\right) \Big\}. \end{aligned} \tag{4.2}$$

It follows from $\mathcal{N} \subset \mathcal{N}'$ that $A' \leq A$. By Sobolev inequality, we see that $A' > 0$. Consider

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2} u + \frac{\alpha\gamma}{p^*} |u|^{\alpha-2} u |v|^\beta, & x \in B(0, R), \\ -\Delta_p v = \mu_2 |v|^{p^*-2} v + \frac{\beta\gamma}{p^*} |u|^\alpha |v|^{\beta-2} v, & x \in B(0, R), \\ u, v \in H_0^1\big(B(0, R)\big), \end{cases} \tag{4.3}$$

where $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\}$. Define

$$\begin{aligned}\mathcal{N}'(R) := \Big\{(u,v) \in H(0,R) \setminus \{(0,0)\} : \int_{B(0,R)} \big(|\nabla u|^p + |\nabla v|^p\big) \\ = \int_{B(0,R)} \big(\mu_1 |u|^{p^*} + \mu_2 |v|^{p^*} + \gamma |u|^\alpha |v|^\beta\big)\Big\}\end{aligned} \tag{4.4}$$

and

$$A'(R) := \inf_{(u,v)\in\mathcal{N}'(R)} I(u,v), \tag{4.5}$$

where $H(0,R) := H_0^1\big(B(0,R)\big) \times H_0^1\big(B(0,R)\big)$. For $\varepsilon \in [0, \min\{\alpha,\beta\}-1)$, consider

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2-2\varepsilon} u + \frac{(\alpha-\varepsilon)\gamma}{p^*-2\varepsilon} |u|^{\alpha-2-\varepsilon} u |v|^{\beta-\varepsilon}, & x \in B(0,1), \\ -\Delta_p v = \mu_2 |v|^{p^*-2-2\varepsilon} v + \frac{(\beta-\varepsilon)\gamma}{p^*-2\varepsilon} |u|^{\alpha-\varepsilon} |v|^{\beta-2-\varepsilon} v, & x \in B(0,1), \\ u, v \in H_0^1\big(B(0,1)\big). \end{cases} \tag{4.6}$$

Define

$$\begin{aligned} I_\varepsilon(u,v) :=& \frac{1}{p} \int_{B(0,1)} \big(|\nabla u|^p + |\nabla v|^p\big) \\ & - \frac{1}{p^*-2\varepsilon} \int_{B(0,1)} \big(\mu_1 |u|^{p^*-2\varepsilon} + \mu_2 |v|^{p^*-2\varepsilon} + \gamma |u|^{\alpha-\varepsilon} |v|^{\beta-\varepsilon}\big), \\ \mathcal{N}'_\varepsilon :=& \Big\{(u,v) \in H(0,1) \setminus \{(0,0)\} : G_\varepsilon(u,v) := \int_{B(0,1)} \big(|\nabla u|^p + |\nabla v|^p\big) \\ & - \int_{B(0,1)} \big(\mu_1 |u|^{p^*-2\varepsilon} + \mu_2 |v|^{p^*-2\varepsilon} + \gamma |u|^{\alpha-\varepsilon} |v|^{\beta-\varepsilon}\big) = 0\Big\}, \end{aligned} \tag{4.7}$$

and

$$A_\varepsilon := \inf_{(u,v)\in\mathcal{N}'_\varepsilon} I_\varepsilon(u,v). \tag{4.8}$$

Lemma 4.1. *Assume that $\frac{2N}{N+2} < p < \frac{N}{2}$, $\alpha, \beta < p$. For $\varepsilon \in (0, \min\{\alpha,\beta\}-1)$, there holds*

$$A_\varepsilon < \min\Big\{ \inf_{(u,0)\in\mathcal{N}'_\varepsilon} I_\varepsilon(u,0), \inf_{(0,v)\in\mathcal{N}'_\varepsilon} I_\varepsilon(0,v) \Big\}.$$

Proof. From $\min\{\alpha,\beta\} \le \frac{p^*}{2}$, it is easy to see that $2 < p^* - 2\varepsilon < p^*$. Then, we may assume that u_i is a least energy solution of

$$-\Delta_p u = \mu_i |u|^{p^*-2-2\varepsilon} u, \quad u \in H_0^1\big(B(0,1)\big), \quad i = 1,2.$$

Therefore,

$$I_\varepsilon(u_1,0) = a_1 := \inf_{(u,0)\in\mathcal{N}'_\varepsilon} I_\varepsilon(u,0), \quad I_\varepsilon(0,u_2) = a_2 := \inf_{(0,v)\in\mathcal{N}'_\varepsilon} I_\varepsilon(0,v).$$

It is claimed that, for any $s \in \mathbb{R}$, there exists an unique $t(s) > 0$ such that $\big(\sqrt[p]{t(s)}u_1, \sqrt[p]{t(s)}su_2\big) \in \mathcal{N}'_\varepsilon$. In fact,

$$\begin{aligned} t(s)^{\frac{p^*-p-2\varepsilon}{p}} &= \frac{\int_{B(0,1)}\big(|\nabla u_1|^p + |s|^p|\nabla u_2|^p\big)}{\int_{B(0,1)}\big(\mu_1|u_1|^{p^*-2\varepsilon} + \mu_2|su_2|^{p^*-2\varepsilon} + \gamma|u_1|^{\alpha-\varepsilon}|su_2|^{\beta-\varepsilon}\big)} \\ &= \frac{qa_1 + qa_2|s|^p}{qa_1 + qa_2|s|^{p^*-2\varepsilon} + |s|^{\beta-\varepsilon}\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}}, \end{aligned}$$

where $q := \frac{p(p^*-2\varepsilon)}{p^*-p-2\varepsilon} = \frac{p(Np-2\varepsilon+2\varepsilon p)}{p^2-2\varepsilon N+2\varepsilon p} \to N$ as $\varepsilon \to 0$. Noticing that $t(0) = 1$, we have

$$\lim_{s\to 0}\frac{t'(s)}{|s|^{\beta-\varepsilon-2}s} = -\frac{(\beta-\varepsilon)\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1},$$

that is,

$$t'(s) = -\frac{(\beta-\varepsilon)\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1}|s|^{\beta-\varepsilon-2}s\big(1+o(1)\big), \quad \text{as } s \to 0.$$

Then,

$$t(s) = 1 - \frac{\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}}{(p^*-2\varepsilon)a_1}|s|^{\beta-\varepsilon}\big(1+o(1)\big), \quad \text{as } s \to 0,$$

and so,

$$t(s)^{\frac{p^*-2\varepsilon}{p}} = 1 - \frac{\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}}{pa_1}|s|^{\beta-\varepsilon}\big(1+o(1)\big), \quad \text{as } s \to 0.$$

Since $\frac{1}{p} - \frac{1}{q} = \frac{1}{p^*-2\varepsilon}$, we have

$$\begin{aligned} A_\varepsilon \le& I_\varepsilon\big(\sqrt[p]{t(s)}u_1, \sqrt[p]{t(s)}su_2\big) \\ =& \Big(\frac{1}{p} - \frac{1}{p^*-2\varepsilon}\Big)\Big(qa_1 + qa_2|s|^{p^*-2\varepsilon} + |s|^{\beta-\varepsilon}\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon}\Big)t^{\frac{p^*-2\varepsilon}{p}} \\ =& a_1 - \Big(\frac{1}{p} - \frac{1}{q}\Big)|s|^{\beta-\varepsilon}\int_{B(0,1)}\gamma|u_1|^{\alpha-\varepsilon}|u_2|^{\beta-\varepsilon} + o(|s|^{\beta-\varepsilon}) \\ <& a_1 = \inf_{(u,0)\in\mathcal{N}'_\varepsilon} I_\varepsilon(u,0) \quad \text{as } |s| \text{ small enough.} \end{aligned}$$

Similarly, $A_\varepsilon < \inf_{(0,v)\in\mathcal{N}'_\varepsilon} I_\varepsilon(0,v)$. This completes the proof. ∎

Noticing the definition of ω_{μ_i} in the proof of Theorem 1.1, similarly as Lemma 4.1, we obtain that

$$\begin{aligned} A' &< \min\Big\{\inf_{(u,0)\in\mathcal{N}'} I(u,0), \inf_{(0,v)\in\mathcal{N}'} I(0,v)\Big\} \\ &= \min\big\{I(\omega_{\mu_1},0), I(0,\omega_{\mu_2})\big\} \\ &= \min\Big\{\frac{1}{N}\mu_1^{-\frac{N-p}{p}}S^{\frac{N}{p}}, \frac{1}{N}\mu_2^{-\frac{N-p}{p}}S^{\frac{N}{p}}\Big\}. \end{aligned} \tag{4.9}$$

Proposition 4.1. *For any $\varepsilon \in (0, \min\{\alpha, \beta\} - 1)$, system (4.6) has a classical positive least energy solution $(u_\varepsilon, v_\varepsilon)$, and $u_\varepsilon, v_\varepsilon$ are radially symmetric decreasing.*

Proof. It is standard to see that $A_\varepsilon > 0$. For $(u, v) \in \mathcal{N}'_\varepsilon$ with $u \geq 0, v \geq 0$, we denote by (u^*, v^*) as its Schwartz symmetrization. By the properties of Schwartz symmetrization and $\gamma > 0$, we get that

$$\int_{B(0,1)} \big(|\nabla u^*|^p + |\nabla v^*|^p\big) \leq \int_{B(0,1)} \big(\mu_1 |u^*|^{p^*-2\varepsilon} + \mu_2 |v^*|^{p^*-2\varepsilon} + \gamma |u^*|^{\alpha-\varepsilon} |v^*|^{\beta-\varepsilon}\big).$$

Obviously, there exists $t^* \in (0, 1]$ such that $\big(\sqrt[p]{t^*} u^*, \sqrt[p]{t^*} v^*\big) \in \mathcal{N}'_\varepsilon$. Therefore,

$$\begin{aligned} I_\varepsilon\big(\sqrt[p]{t^*} u^*, \sqrt[p]{t^*} v^*\big) &= \Big(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\Big) t^* \int_{B(0,1)} \big(|\nabla u^*|^p + |\nabla v^*|^p\big) \\ &\leq \frac{p^* - 2\varepsilon - p}{p(p^* - 2\varepsilon)} \int_{B(0,1)} \big(|\nabla u|^p + |\nabla v|^p\big) \\ &= I_\varepsilon(u, v). \end{aligned} \tag{4.10}$$

Then, we may choose a minimizing sequence $(u_n, v_n) \in \mathcal{N}'_\varepsilon$ of A_ε such that $(u_n, v_n) = (u_n^*, v_n^*)$ and $I_\varepsilon(u_n, v_n) \to A_\varepsilon$ as $n \to \infty$. By (4.10), we see that u_n, v_n are uniformly bounded in $H_0^1\big(B(0,1)\big)$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u_\varepsilon, v_n \rightharpoonup v_\varepsilon$ weakly in $H_0^1\big(B(0,1)\big)$. Since $H_0^1\big(B(0,1)\big) \hookrightarrow\hookrightarrow L^{p^*-2\varepsilon}\big(B(0,1)\big)$, we deduce that

$$\begin{aligned} &\int_{B(0,1)} \big(\mu_1 |u_\varepsilon|^{p^*-2\varepsilon} + \mu_2 |v_\varepsilon|^{p^*-2\varepsilon} + \gamma |u_\varepsilon|^{\alpha-\varepsilon} |v_\varepsilon|^{\beta-\varepsilon}\big) \\ &= \lim_{n\to\infty} \int_{B(0,1)} \big(\mu_1 |u_n|^{p^*-2\varepsilon} + \mu_2 |v_n|^{p^*-2\varepsilon} + \gamma |u_n|^{\alpha-\varepsilon} |v_n|^{\beta-\varepsilon}\big) \\ &= \frac{p(p^* - 2\varepsilon)}{p^* - 2\varepsilon - p} \lim_{n\to\infty} I_\varepsilon(u_n, v_n) \\ &= \frac{p(p^* - 2\varepsilon)}{p^* - 2\varepsilon - p} A_\varepsilon > 0, \end{aligned}$$

which implies that $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$. Moreover, $u_\varepsilon \geq 0, v_\varepsilon \geq 0$ are radially symmetric. Noticing that $\int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big) \leq \lim_{n\to\infty} \int_{B(0,1)} \big(|\nabla u_n|^p + |\nabla v_n|^p\big)$, we get that

$$\int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big) \leq \int_{B(0,1)} \big(\mu_1 |u_\varepsilon|^{p^*-2\varepsilon} + \mu_2 |v_\varepsilon|^{p^*-2\varepsilon} + \gamma |u_\varepsilon|^{\alpha-\varepsilon} |v_\varepsilon|^{\beta-\varepsilon}\big).$$

Then, there exists $t_\varepsilon \in (0, 1]$ such that $\big(\sqrt[p]{t_\varepsilon} u_\varepsilon, \sqrt[p]{t_\varepsilon} v_\varepsilon\big) \in \mathcal{N}'_\varepsilon$, and therefore,

$$\begin{aligned} A_\varepsilon &\leq I_\varepsilon\big(\sqrt[p]{t_\varepsilon} u_\varepsilon, \sqrt[p]{t_\varepsilon} v_\varepsilon\big) \\ &= \Big(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\Big) t_\varepsilon \int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big) \\ &\leq \lim_{n\to\infty} \frac{p^* - 2\varepsilon - p}{p(p^* - 2\varepsilon)} \int_{B(0,1)} \big(|\nabla u_n|^p + |\nabla v_n|^p\big) \\ &= \lim_{n\to\infty} I_\varepsilon(u_n, v_n) = A_\varepsilon, \end{aligned}$$

which yields that $t_\varepsilon = 1$, $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}'_\varepsilon$, $I(u_\varepsilon, v_\varepsilon) = A_\varepsilon$, and

$$\int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big) = \lim_{n\to\infty} \int_{B(0,1)} \big(|\nabla u_n|^p + |\nabla v_n|^p\big).$$

That is, $u_n \to u_\varepsilon, v_n \to v_\varepsilon$ strongly in $H_0^1\big(B(0,1)\big)$. It follows from the standard minimization theory that there exists a Lagrange multiplier $L \in \mathbb{R}$ satisfying

$$I'_\varepsilon(u_\varepsilon, v_\varepsilon) + LG'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0.$$

Since $I'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = G_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ and

$$\begin{aligned} &G'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) \\ &= -(p^* - 2\varepsilon - p) \int_{B(0,1)} \big(\mu_1 |u_\varepsilon|^{p^*-2\varepsilon} + \mu_2 |v_\varepsilon|^{p^*-2\varepsilon} + \gamma |u_\varepsilon|^{\alpha-\varepsilon} |v_\varepsilon|^{\beta-\varepsilon}\big) < 0, \end{aligned}$$

we get that $L = 0$ and so $I'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$. By $A_\varepsilon = I(u_\varepsilon, v_\varepsilon)$ and Lemma 4.1, we have $u_\varepsilon \not\equiv 0$ and $v_\varepsilon \not\equiv 0$. Since $u_\varepsilon, v_\varepsilon \geq 0$ are radially symmetric decreasing, by the regularity theory and the maximum principle, we obtain that $(u_\varepsilon, v_\varepsilon)$ is a classical positive least energy solution of (4.6). This completes the proof. □

Proof of Theorem 1.3. We claim that

$$A'(R) \equiv A' \ \text{ for all } R > 0. \tag{4.11}$$

Indeed, assume $R_1 < R_2$. Since $\mathcal{N}'(R_1) \subset \mathcal{N}'(R_2)$, we get that $A'(R_2) \leq A'(R_1)$. On the other hand, for every $(u, v) \in \mathcal{N}'(R_2)$, define

$$\big(u_1(x), v_1(x)\big) := \Big(\Big(\frac{R_2}{R_1}\Big)^{\frac{N-p}{p}} u\Big(\frac{R_2}{R_1}x\Big), \Big(\frac{R_2}{R_1}\Big)^{\frac{N-p}{p}} v\Big(\frac{R_2}{R_1}x\Big)\Big),$$

then it is easy to see that $(u_1, v_1) \in \mathcal{N}'(R_1)$. Thus, we have

$$A'(R_1) \leq I(u_1, v_1) = I(u, v), \ \ \forall (u, v) \in \mathcal{N}'(R_2),$$

which means that $A'(R_1) \leq A'(R_2)$. Hence, $A'(R_1) = A'(R_2)$. Obviously, $A' \leq A'(R)$. Let $(u_n, v_n) \in \mathcal{N}'$ be a minimizing sequence of A'. We may assume that $u_n, v_n \in H_0^1\big(B(0, R_n)\big)$ for some $R_n > 0$. Therefore, $(u_n, v_n) \in \mathcal{N}'(R_n)$ and

$$A' = \lim_{n\to\infty} I(u_n, v_n) \geq \lim_{n\to\infty} A'(R_n) = A'(R),$$

which completes the proof of the claim.

Recalling (4.4) and (4.7), for every $(u, v) \in \mathcal{N}'(1)$, there exists $t_\varepsilon > 0$ with $t_\varepsilon \to 1$ as $\varepsilon \to 0$ such that $\big(\sqrt[p]{t_\varepsilon} u, \sqrt[p]{t_\varepsilon} v\big) \in \mathcal{N}'_\varepsilon$. Then,

$$\limsup_{\varepsilon\to 0} A_\varepsilon \leq \limsup_{\varepsilon\to 0} I_\varepsilon\big(\sqrt[p]{t_\varepsilon} u, \sqrt[p]{t_\varepsilon} v\big) = I(u, v), \ \ \forall (u, v) \in \mathcal{N}'(1).$$

It follows from (4.11) that

$$\limsup_{\varepsilon\to 0} A_\varepsilon \leq A'(1) = A'. \tag{4.12}$$

According to Proposition 4.1, we may let $(u_\varepsilon, v_\varepsilon)$ be a positive least energy solution of (4.6), which is radially symmetric decreasing. By (4.7) and Sobolev inequality, we have

$$A_\varepsilon = \frac{p^* - 2\varepsilon - 2}{2(p^* - 2\varepsilon)} \int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big) \geq C > 0, \ \ \forall \varepsilon \in \big(0, \frac{\min\{\alpha,\beta\} - 1}{2}\big], \tag{4.13}$$

where C is independent of ε. Then, it follows from (4.12) that $u_\varepsilon, v_\varepsilon$ are uniformly bounded in $H_0^1\big(B(0,1)\big)$. We may assume that $u_\varepsilon \rightharpoonup u_0, v_\varepsilon \rightharpoonup v_0$, up to a subsequence, weakly in $H_0^1\big(B(0,1)\big)$. Hence, (u_0, v_0) is a solution of

$$\begin{cases} -\Delta_p u = \mu_1 |u|^{p^*-2}u + \frac{\alpha\gamma}{p^*}|u|^{\alpha-2}u|v|^\beta, \ \ x \in B(0,1), \\ -\Delta_p v = \mu_2 |v|^{p^*-2}v + \frac{\beta\gamma}{p^*}|u|^{\alpha}|v|^{\beta-2}v, \ \ x \in B(0,1), \\ u, v \in H_0^1\big(B(0,1)\big). \end{cases} \tag{4.14}$$

Suppose by contradiction that $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty$ is uniformly bounded. Then, by the Dominated Convergent Theorem, we get that

$$\lim_{\varepsilon\to 0} \int_{B(0,1)} u_\varepsilon^{p^*-2\varepsilon} = \int_{B(0,1)} u_0^{p^*}, \qquad \lim_{\varepsilon\to 0} \int_{B(0,1)} v_\varepsilon^{p^*-2\varepsilon} = \int_{B(0,1)} v_0^{p^*},$$
$$\lim_{\varepsilon\to 0} \int_{B(0,1)} u_\varepsilon^{\alpha-\varepsilon} v_\varepsilon^{\beta-\varepsilon} = \int_{B(0,1)} u_0^{\alpha} v_0^{\beta}.$$

Combining these with $I_\varepsilon'(u_\varepsilon, v_\varepsilon) = I'(u_0, v_0)$, similarly as the proof of Proposition 4.1, we see that $u_\varepsilon \to u_0, v_\varepsilon \to v_0$ strongly in $H_0^1\big(B(0,1)\big)$. It follows from (4.13) that $(u_0, v_0) \neq (0,0)$, and moreover, $u_0 \geq 0, v_0 \geq 0$. Without loss of generality, we may assume that $u_0 \not\equiv 0$. By the strong maximum principle, we obtain that $u_0 > 0$ in $B(0,1)$. By Pohozaev identity, we have a contradiction

$$0 < \int_{\partial B(0,1)} \big(|\nabla u_0|^p + |\nabla v_0|^p\big)(x \cdot \nu) d\sigma = 0,$$

where ν is the outward unit normal vector on $\partial B(0,1)$. Hence, $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \to \infty$, as $\varepsilon \to 0$. Let $K_\varepsilon := \max\{u_\varepsilon(0), v_\varepsilon(0)\}$. Since $u_\varepsilon(0) = \max_{B(0,1)} u_\varepsilon(x)$ and $v_\varepsilon(0) = \max_{B(0,1)} v_\varepsilon(x)$, we see that $K_\varepsilon \to +\infty$, as $\varepsilon \to 0$. Setting

$$U_\varepsilon(x) := K_\varepsilon^{-1} u_\varepsilon(K_\varepsilon^{-a_\varepsilon} x), \quad V_\varepsilon(x) := K_\varepsilon^{-1} v_\varepsilon(K_\varepsilon^{-a_\varepsilon} x), \quad a_\varepsilon := \frac{p^* - p - p\varepsilon}{p}.$$

we have

$$\max\{U_\varepsilon(0), V_\varepsilon(0)\} = \max\Big\{ \max_{x\in B(0,K_\varepsilon^{a_\varepsilon})} U_\varepsilon(x), \max_{x\in B(0,K_\varepsilon^{a_\varepsilon})} V_\varepsilon(x) \Big\} = 1 \tag{4.15}$$

and $(U_\varepsilon, V_\varepsilon)$ is a solution of

$$\begin{cases} -\Delta_p U_\varepsilon = \mu_1 U_\varepsilon^{p^*-2\varepsilon-1} + \frac{(\alpha-\varepsilon)\gamma}{p^*-2\varepsilon} U_\varepsilon^{\alpha-1-\varepsilon} V_\varepsilon^{\beta-\varepsilon}, \ \ x \in B(0, K_\varepsilon^{a_\varepsilon}), \\ -\Delta_p V_\varepsilon = \mu_2 V_\varepsilon^{p^*-2\varepsilon-1} + \frac{(\beta-\varepsilon)\gamma}{p^*-2\varepsilon} U_\varepsilon^{\alpha-\varepsilon} V_\varepsilon^{\beta-1-\varepsilon}, \ \ x \in B(0, K_\varepsilon^{a_\varepsilon}). \end{cases}$$

Since

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon(x)|^p \mathrm{d}x = K_\varepsilon^{a_\varepsilon(N-p)-p} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(y)|^p \mathrm{d}y$$
$$= K_\varepsilon^{-(N-p)\varepsilon} \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^p \mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|^p \mathrm{d}x,$$

we see that $\{(U_\varepsilon, V_\varepsilon)\}_{n\ge 1}$ is bounded in D. By elliptic estimates, we get that, up to a subsequence, $(U_\varepsilon, V_\varepsilon) \to (U, V) \in D$ uniformly in every compact subset of $\mathbb{R}^N$ as $\varepsilon \to 0$, and (U, V) is a solution of (1.1), that is, $I'(U, V) = 0$. Moreover, $U \ge 0, V \ge 0$ are radially symmetric decreasing. By (4.15), we have $(U, V) \ne (0, 0)$ and so $(U, V) \in \mathcal{N}'$. Thus,

$$\begin{aligned}
A' \le I(U,V) &= \Big(\frac{1}{p} - \frac{1}{p^*}\Big) \int_{\mathbb{R}^N} \big(|\nabla U|^p + |\nabla V|^p\big)\mathrm{d}x \\
&\le \liminf_{\varepsilon\to 0} \Big(\frac{1}{p} - \frac{1}{p^*}\Big) \int_{B(0,K_\varepsilon^{a_\varepsilon})} \big(|\nabla U_\varepsilon|^p + |\nabla V_\varepsilon|^p\big)\mathrm{d}x \\
&= \liminf_{\varepsilon\to 0} \Big(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\Big) \int_{B(0,K_\varepsilon^{a_\varepsilon})} \big(|\nabla U_\varepsilon|^p + |\nabla V_\varepsilon|^p\big)\mathrm{d}x \\
&\le \liminf_{\varepsilon\to 0} \Big(\frac{1}{p} - \frac{1}{p^* - 2\varepsilon}\Big) \int_{B(0,1)} \big(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\big)\mathrm{d}x \\
&= \liminf_{\varepsilon\to 0} A_\varepsilon.
\end{aligned}$$

It follows from (4.12) that $A' \le I(U, V) \le \liminf_{\varepsilon\to 0} A_\varepsilon \le A'$, which means that $I(U, V) = A'$. By (4.9), we get that $U \not\equiv 0$ and $V \not\equiv 0$. The strong maximum principle guarantees that $U > 0$ and $V > 0$. Since $(U, V) \in \mathcal{N}$, we have $I(U, V) \ge A \ge A'$. Therefore,

$$I(U, V) = A = A', \tag{4.16}$$

that is, (U, V) is a positive least energy solution of (1.1) with (H_1) holding, which is radially symmetric decreasing. This completes the proof. □

Remark 4.1. *If (H_1) and (C_2) hold, then it can be seen from Theorems 1.2 and 1.3 that $(\sqrt[p]{k_0}U_{\varepsilon,y}, \sqrt[p]{l_0}U_{\varepsilon,y})$ is a positive least energy solution of (1.1), where (k_0, l_0) is defined by (1.10) and $U_{\varepsilon,y}$ is defined by (1.4).*

Proof of Theorem 1.4. To prove the existence of $\big(k(\gamma), l(\gamma)\big)$ for $\gamma > 0$ small, recalling (3.2), we denote $F_i(k, l)$ by $F_i(k, l, \gamma),\ i = 1, 2$ in this proof. Let $k(0) = \mu_1^{-\frac{p}{p^*-p}}$ and $l(0) = \mu_2^{-\frac{p}{p^*-p}}$. Then $F_1\big(k(0), l(0), 0\big) = F_2\big(k(0), l(0), 0\big) = 0$. Obviously, we have

$$\partial_k F_1\big(k(0), l(0), 0\big) = \frac{p^* - p}{p}\mu_1 k^{\frac{p^*-2p}{p}} > 0,$$
$$\partial_l F_1\big(k(0), l(0), 0\big) = \partial_k F_2\big(k(0), l(0), 0\big) = 0,$$
$$\partial_l F_2\big(k(0), l(0), 0\big) = \frac{p^* - p}{p}\mu_2 l^{\frac{p^*-2p}{p}} > 0,$$

which implies that

$$\det\begin{pmatrix} \partial_k F_1\big(k(0),l(0),0\big) & \partial_l F_1\big(k(0),l(0),0\big) \\ \partial_k F_2\big(k(0),l(0),0\big) & \partial_l F_2\big(k(0),l(0),0\big) \end{pmatrix} > 0.$$

By the implicit function theorem, we see that $k(\gamma), l(\gamma)$ are well defined and of class C^1 in $(-\gamma_2, \gamma_2)$ for some $\gamma_2 > 0$, and $F_1\big(k(\gamma), l(\gamma), \gamma\big) = F_2\big(k(\gamma), l(\gamma), \gamma\big) = 0$. Then, $\big(\sqrt[p]{k(\gamma)}U_{\varepsilon,y},\, \sqrt[p]{l(\gamma)}U_{\varepsilon,y}\big)$ is a positive solution of (1.1). Noticing that

$$\lim_{\gamma\to 0}\big(k(\gamma)+l(\gamma)\big) = k(0)+l(0) = \mu_1^{-\frac{N-p}{p}} + \mu_2^{-\frac{N-p}{p}},$$

there exists $\gamma_1 \in (0, \gamma_2]$ such that

$$k(\gamma)+l(\gamma) > \min\big\{\mu_1^{-\frac{N-p}{p}},\, \mu_2^{-\frac{N-p}{p}}\big\}, \ \ \forall \gamma \in (0, \gamma_1).$$

It follows from (4.9) and (4.16) that

$$\begin{aligned} I\big(\sqrt[p]{k(\gamma)}U_{\varepsilon,y},\, \sqrt[p]{l(\gamma)}U_{\varepsilon,y}\big) &= \frac{1}{N}\big(k(\gamma)+l(\gamma)\big)S^{\frac{N}{p}} \\ &> \min\Big\{\frac{1}{N}\mu_1^{-\frac{N-p}{p}}S^{\frac{N}{p}}, \frac{1}{N}\mu_2^{-\frac{N-p}{p}}S^{\frac{N}{p}}\Big\} \\ &> A' = A = I(U,V), \end{aligned}$$

that is, when (H_1) is satisfied, $\big(\sqrt[p]{k(\gamma)}U_{\varepsilon,y},\, \sqrt[p]{l(\gamma)}U_{\varepsilon,y}\big)$ is a different positive solution of (1.1) with respect to (U,V). This completes the proof. □

5 Proof of Theorem 1.5

In this section, we consider the case (H_2).

Proposition 5.1. *Let $q, r > 1$ satisfy $q + r \le p^*$ and set*

$$\begin{aligned} S_{q,r}(\Omega) &= \inf_{\substack{u,v\in W_0^{1,p}(\Omega)\\ u,v\neq 0}} \frac{\int_\Omega \left(|\nabla u|^p + |\nabla v|^p\right)dx}{\left(\int_\Omega |u|^q\,|v|^r\,dx\right)^{\frac{p}{q+r}}}, \\ S_{q+r}(\Omega) &= \inf_{\substack{u\in W_0^{1,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^p\,dx}{\left(\int_\Omega |u|^{q+r}\,dx\right)^{\frac{p}{q+r}}}. \end{aligned}$$

Then

$$S_{q,r}(\Omega) = \frac{q+r}{\left(q^q\,r^r\right)^{\frac{1}{q+r}}}\,S_{q+r}(\Omega). \tag{5.1}$$

Moreover, if u_0 is a minimizer for $S_{q+r}(\Omega)$, then $(q^{\frac{1}{p}}\,u_0, r^{\frac{1}{p}}\,u_0)$ is a minimizer for $S_{q,r}(\Omega)$.

Proof. For $u \neq 0$ in $W_0^{1,p}(\Omega)$ and $t > 0$, taking $v = t^{-\frac{1}{p}}\, u$ in the first quotient gives

$$S_{q,r}(\Omega) \le \Big[t^{\frac{r}{q+r}} + t^{-\frac{q}{q+r}}\Big]\, \frac{\int_\Omega |\nabla u|^p\, dx}{\big(\int_\Omega |u|^{q+r}\, dx\big)^{\frac{p}{q+r}}},$$

and minimizing the right-hand side over u and t shows that $S_{q,r}(\Omega)$ is less than or equal to the right-hand side of (5.1). For $u, v \neq 0$ in $W_0^{1,p}(\Omega)$, let $w = t^{\frac{1}{p}}\, v$, where

$$t^{\frac{q+r}{p}} = \frac{\int_\Omega |u|^{q+r}\, dx}{\int_\Omega |v|^{q+r}\, dx}.$$

Then $\int_\Omega |u|^{q+r}\, dx = \int_\Omega |w|^{q+r}\, dx$ and hence

$$\int_\Omega |u|^q\, |w|^r\, dx \le \int_\Omega |u|^{q+r}\, dx = \int_\Omega |w|^{q+r}\, dx$$

by the Hölder inequality, so

$$\begin{aligned}
&\frac{\int_\Omega \left(|\nabla u|^p + |\nabla v|^p\right) dx}{\big(\int_\Omega |u|^q\, |v|^r\, dx\big)^{\frac{p}{q+r}}}\\
&= \frac{\int_\Omega \Big(t^{\frac{r}{q+r}}\, |\nabla u|^p + t^{-\frac{q}{q+r}}\, |\nabla w|^p\Big)\, dx}{\big(\int_\Omega |u|^q\, |w|^r\, dx\big)^{\frac{p}{q+r}}}\\
&\ge t^{\frac{r}{q+r}}\, \frac{\int_\Omega |\nabla u|^p\, dx}{\big(\int_\Omega |u|^{q+r}\, dx\big)^{\frac{p}{q+r}}} + t^{-\frac{q}{q+r}}\, \frac{\int_\Omega |\nabla w|^p\, dx}{\big(\int_\Omega |w|^{q+r}\, dx\big)^{\frac{p}{q+r}}}\\
&\ge \Big[t^{\frac{r}{q+r}} + t^{-\frac{q}{q+r}}\Big]\, S_{q+r}(\Omega).
\end{aligned}$$

The last expression is greater than or equal to the right-hand side of (5.1), so minimizing over (u, v) gives the reverse inequality. $\square$

By Proposition 5.1,

$$S_{a,b}(\Omega) = \frac{p}{(a^a\, b^b)^{\frac{1}{p}}}\, \lambda_1(\Omega), \qquad S_{\alpha,\beta} = \frac{p^*}{(\alpha^\alpha\, \beta^\beta)^{\frac{1}{p^*}}}\, S, \tag{5.2}$$

where $\lambda_1(\Omega) > 0$ is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω. When (H_2) is satisfied, we will obtain a nontrivial nonnegative solution of system (1.1) for $\lambda < S_{a,b}(\Omega)$. Consider the C^1-functional

$$\Phi(w) = \frac{1}{p} \int_\Omega \left[|\nabla u|^p + |\nabla v|^p - \lambda (u^+)^a (v^+)^b\right] dx - \frac{1}{p^*} \int_\Omega (u^+)^\alpha (v^+)^\beta dx, \quad w \in W,$$

where $W = D_0^{1,p}(\Omega) \times D_0^{1,p}(\Omega)$ with the norm given by $\|w\|^p = |\nabla u|_p^p + |\nabla v|_p^p$ for $w = (u, v)$, $|\cdot|_p$ denotes the norm in $L^p(\Omega)$, and $u^\pm(x) = \max\{\pm u(x), 0\}$ are the positive and negative parts of u, respectively. If w is a critical point of Φ,

$$0 = \Phi'(w)\, (u^-, v^-) = \int_\Omega \left(|\nabla u^-|^p + |\nabla v^-|^p\right) dx$$

and hence $(u^-, v^-) = 0$, so $w = (u^+, v^+)$ is a nonnegative weak solution of (1.1) with (H_2) holding.

Proposition 5.2. *If $0 \neq c < \frac{S_{\alpha,\beta}^{\frac{N}{p}}}{N}$ and $\lambda < S_{a,b}(\Omega)$, then every $(PS)_c$ sequence of Φ has a subsequence that converges weakly to a nontrivial critical point of Φ.*

Proof. Let $\{w_j\}$ be a $(PS)_c$ sequence. Then

$$\begin{aligned}\Phi(w_j) =&\frac{1}{p}\int_\Omega \left[|\nabla u_j|^p + |\nabla v_j|^p - \lambda\,(u_j^+)^a\,(v_j^+)^b\right] dx - \frac{1}{p^*}\int_\Omega (u_j^+)^\alpha\,(v_j^+)^\beta\,dx\\ =&c + o(1)\end{aligned}$$

and

$$\begin{aligned}\Phi'(w_j)\,w_j =& \int_\Omega \left[|\nabla u_j|^p + |\nabla v_j|^p - \lambda\,(u_j^+)^a\,(v_j^+)^b\right] dx - \int_\Omega (u_j^+)^\alpha\,(v_j^+)^\beta\,dx\\ =&o(\|w_j\|),\end{aligned} \tag{5.3}$$

so

$$\frac{1}{N}\int_\Omega \left[|\nabla u_j|^p + |\nabla v_j|^p - \lambda\,(u_j^+)^a\,(v_j^+)^b\right] dx = c + o(\|w_j\| + 1). \tag{5.4}$$

Since the integral on the left is greater than or equal to $(1 - \frac{\lambda}{S_{a,b}(\Omega)})\|w_j\|^p$, $\lambda < S_{a,b}(\Omega)$, and $p > 1$, it follows that $\{w_j\}$ is bounded in W. So a renamed subsequence converges to some w weakly in W, strongly in $L^s(\Omega) \times L^t(\Omega)$ for all $1 \leq s, t < p^*$, and a.e. in Ω. Then $w_j \to w$ strongly in $W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega)$ for all $1 \leq q, r < p$ by Boccardo and Murat [6, Theorem 2.1], and hence $\nabla w_j \to \nabla w$ a.e. in Ω for a further subsequence. It then follows that w is a critical point of Φ.

Suppose $w = 0$. Since $\{w_j\}$ is bounded in W and converges to zero in $L^p(\Omega) \times L^p(\Omega)$, (5.3) and the Hölder inequality gives

$$o(1) = \int_\Omega (|\nabla u_j|^p + |\nabla v_j|^p)\,dx - \int_\Omega (u_j^+)^\alpha\,(v_j^+)^\beta\,dx \geq \|w_j\|^p\left(1 - \frac{\|w_j\|^{p^*-p}}{S_{\alpha,\beta}^{\frac{p^*}{p}}}\right).$$

If $\|w_j\| \to 0$, then $\Phi(w_j) \to 0$, contradicting $c \neq 0$, so this implies

$$\|w_j\|^p \geq S_{\alpha,\beta}^{\frac{N}{p}} + o(1)$$

for a renamed subsequence. Then (5.4) gives

$$c = \frac{\|w_j\|^p}{N} + o(1) \geq \frac{S_{\alpha,\beta}^{\frac{N}{p}}}{N} + o(1),$$

contradicting $c < \frac{S_{\alpha,\beta}^{\frac{N}{p}}}{N}$. □

Recalling (1.4) and (1.5), let $\eta : [0, \infty) \to [0, 1]$ be a smooth cut-off function such that $\eta(s) = 1$ for $s \le \frac{1}{4}$ and $\eta(s) = 0$ for $s \ge \frac{1}{2}$, and set

$$u_{\varepsilon,\rho}(x) = \eta\biggl(\frac{|x|}{\rho}\biggr)\, U_{\varepsilon,0}(x)$$

for $\rho > 0$. We have the following estimates for $u_{\varepsilon,\rho}$ (see [15, Lemma 3.1]):

$$\int_{\mathbb{R}^N} |\nabla u_{\varepsilon,\rho}|^p\, dx \le S^{\frac{N}{p}} + C\left(\frac{\varepsilon}{\rho}\right)^{\frac{N-p}{p-1}}, \tag{5.5}$$

$$\int_{\mathbb{R}^N} u_{\varepsilon,\rho}^p\, dx \ge \begin{cases} \dfrac{1}{C}\, \varepsilon^p\, \log\biggl(\dfrac{\rho}{\varepsilon}\biggr) - C\, \varepsilon^p, & N = p^2, \\ \dfrac{1}{C}\, \varepsilon^p - C\, \rho^p \left(\dfrac{\varepsilon}{\rho}\right)^{\frac{N-p}{p-1}}, & N > p^2, \end{cases} \tag{5.6}$$

$$\int_{\mathbb{R}^N} u_{\varepsilon,\rho}^{p^*}\, dx \ge S^{\frac{N}{p}} - C\left(\frac{\varepsilon}{\rho}\right)^{\frac{N}{p-1}}, \tag{5.7}$$

where $C = C(N, p)$. We will make use of these estimates in the proof of our last theorem.

Proof of Theorem 1.5. In view of (5.2),

$$\Phi(w) \ge \frac{1}{p}\left(1 - \frac{\lambda}{S_{a,b}(\Omega)}\right)\|w\|^p - \frac{1}{p^*\, S_{\alpha,\beta}^{\frac{p^*}{p}}}\, \|w\|^{p^*},$$

so the origin is a strict local minimizer of Φ. We may assume without loss of generality that $0 \in \Omega$. Fix $\rho > 0$ so small that $\Omega \supset B_\rho(0) \supset \operatorname{supp} u_{\varepsilon,\rho}$, and let $w_\varepsilon = (\alpha^{\frac{1}{p}}\, u_{\varepsilon,\rho}, \beta^{\frac{1}{p}}\, u_{\varepsilon,\rho}) \in W$. Noting that

$$\begin{aligned} \Phi(Rw_\varepsilon) &= \frac{R^p}{p}\left(p^* |\nabla u_{\varepsilon,\rho}|_p^p - \lambda \alpha^{\frac{a}{p}} \beta^{\frac{b}{p}} |u_{\varepsilon,\rho}|_p^p\right) - \frac{R^{p^*}}{p^*}\, \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} |u_{\varepsilon,\rho}|_{p^*}^{p^*} \\ &\to -\infty \end{aligned}$$

as $R \to +\infty$, fix $R_0 > 0$ so large that $\Phi(R_0 w_\varepsilon) < 0$. Then let

$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0,\ \gamma(1) = R_0 w_\varepsilon\}$$

and set

$$c := \inf_{\gamma \in \Gamma}\ \max_{t \in [0,1]}\ \Phi(\gamma(t)) > 0.$$

By the mountain pass theorem, Φ has a $(PS)_c$ sequence $\{w_j\}$.

Since $t \mapsto tR_0 w_\varepsilon$ is a path in Γ,

$$c \le \max_{t \in [0,1]}\ \Phi(tR_0 w_\varepsilon) = \frac{1}{N}\left(\frac{p^* |\nabla u_{\varepsilon,\rho}|_p^p - \lambda (\alpha^a \beta^b)^{\frac{1}{p}} |u_{\varepsilon,\rho}|_p^p}{(\alpha^\alpha \beta^\beta)^{\frac{1}{p^*}} |u_{\varepsilon,\rho}|_{p^*}^p}\right)^{\frac{N}{p}} =: \frac{1}{N}\, S_\varepsilon^{\frac{N}{p}}. \tag{5.8}$$

By (5.5)–(5.7),

$$
\begin{aligned}
S_\varepsilon \leq & \frac{p^*\, S^p + \dfrac{\lambda(\alpha^a \beta^b)^{\frac{1}{p}}}{C}\; \varepsilon^p\, \log\varepsilon + O(\varepsilon^p)}{(\alpha^\alpha \beta^\beta)^{\frac{1}{p^*}} \left(S^p + O(\varepsilon^{\frac{p^2}{p-1}})\right)^{\frac{p-1}{p}}} \\
= & S_{\alpha,\beta} - \left(\frac{\lambda \alpha^{\frac{a}{p}-\frac{\alpha}{p^*}} \beta^{\frac{b}{p}-\frac{\beta}{p^*}}}{C S^{p-1}} |\log\varepsilon| + O(1)\right) \varepsilon^p
\end{aligned}
$$

if $N = p^2$ and

$$
\begin{aligned}
S_\varepsilon \leq & \frac{p^*\, S^{\frac{N}{p}} - \dfrac{\lambda(\alpha^a \beta^b)^{\frac{1}{p}}}{C}\; \varepsilon^p + O(\varepsilon^{\frac{N-p}{p-1}})}{(\alpha^\alpha \beta^\beta)^{\frac{1}{p^*}} \left(S^{\frac{N}{p}} + O(\varepsilon^{\frac{N}{p-1}})\right)^{\frac{N-p}{N}}} \\
= & S_{\alpha,\beta} - \left(\frac{\lambda \alpha^{\frac{a}{p}-\frac{\alpha}{p^*}} \beta^{\frac{b}{p}-\frac{\beta}{p^*}}}{C S^{\frac{N-p}{p}}} + O(\varepsilon^{\frac{N-p^2}{p-1}})\right) \varepsilon^p
\end{aligned}
$$

if $N > p^2$, so $S_\varepsilon < S_{\alpha,\beta}$ if $\varepsilon > 0$ is sufficiently small. So $c < \frac{S_{\alpha,\beta}^{\frac{N}{p}}}{N}$ by (5.8), and hence a subsequence of $\{w_j\}$ converges weakly to a nontrivial critical point of Φ by Proposition 5.2, which then is a nontrivial nonnegative solution of (1.1) with (H_2) holding. $\square$

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